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Correlators of Short Multi-Trace Operators in $\mathcal{N} = 4$ Supersymmetric Yang-Mills

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Abstract

Multi-trace scalar operators in the $(0, k, 0)$ representations of $SU(4)_R$ share many properties with their single-trace analogs. These multi-trace operators are primary fields of short representations of the $\mathcal{N} = 4$ superconformal group. Thus, they have protected dimensions. We show that two- and three-point correlator functions of such operators do not receive radiative corrections at order g^2 in perturbation theory for an arbitrary gauge group.

1 Introduction

Short representations of $\mathcal{N} = 4$ superconformal algebra play an important role in the correspondence between supersymmetric Yang-Mills (SYM) theory and supergravity in anti-de Sitter (AdS) space [1–3]. Operators in the short representations have protected dimensions, while the dimensions of the long representations depend on the coupling constant. Short operators with a single trace over the gauge indices are believed to correspond to single-particle states of supergravity in the AdS background. Multi-trace operators relate to multi-particle states. The long operators with single traces correspond to string states.

Much work has been devoted to the investigations of correlator functions using either the supergravity dual [2–5] or directly computing in perturbation theory in SYM [6–9]. For additional references see the review article [10]. Since the supergravity calculations are applicable for large values of the 't Hooft coupling, they cannot be easily compared with the weak-coupling perturbation theory. Therefore, it is interesting to study non-renormalized quantities. If a free-field value of some quantity coincides with its large-coupling value obtained by supergravity calculations, then such a result can be regarded as a hint for a non-renormalization theorem. For example, two- and three-point functions of single-trace short operators were found to be equal to their free-field values [5]. By checking if such non-renormalization theorems hold at low orders of perturbation theory one can test the conjectured duality. Perturbative calculations in field theory also provide information beyond the leading order in the $1/N$ expansion. See Refs. [6–8, 11] for calculations of two- and three-point functions. Of course, there should be a symmetry reason for a non-renormalization theorem to hold to all orders of perturbation theory and even non-perturbatively [12–14].

In addition to single-trace short multiplets there are also multi-trace operators in short representations of the $\mathcal{N} = 4$ superconformal group. This fact is certainly known, but not widely appreciated. See Refs. [15, 16] for recent discussions on multi-trace operators. The main result of this paper is that two- and three-point functions of such short multi-trace operators are not renormalized at order g^2 in perturbation theory, independent of the gauge group. Multi-trace operators are in short representations whenever they transform in the same representations of the flavor symmetry as the single-trace short operators. That is, a primary operator proportional to k powers of the scalar field is short if it transforms in the k -index traceless symmetric representation of $SO(6)_R$, or equivalently in $(0, k, 0)$ of $SU(4)_R$. Two facts underline the relevance of short multi-trace operators for the Maldacena duality. First, single- and multi-trace operators have non-vanishing two-point functions already at the tree level.* Second, the supergravity dual is blind to the gauge structure of $\mathcal{N} = 4$ SYM theory since only the gauge-invariant operators enter the correspondence. It is therefore possible that at sub-leading orders in N , the true correspondence between the single-particle states of supergravity and the short representations of superconformal $\mathcal{N} = 4$ theory involves

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a linear combination of single- and multi-trace operators. After all, such operators have identical quantum numbers.

We first review why multi-trace operators in the $(0, k, 0)$ representations of $SU(4)_R$ are indeed short and why their dimensions are protected. Next, we show that two-point functions of arbitrary linear combinations of operators in the $(0, k, 0)$, either single- or multiple-trace are not corrected at order g^2 in perturbation theory. Our results are an extension of Ref. [6], we show that the techniques developed there are also applicable to multi-trace operators. We follow the notation of Ref. [6], where it was shown that two- and three-point functions of short single-trace operators are not renormalized at $O(g^2)$. Afterwards, we repeat the calculation for a particularly simple case of a three-point function, leaving for the Appendix the proof that arbitrary three-point functions of linear combinations of short operators are not renormalized at $O(g^2)$.

$\mathcal{N} = 4$ supersymmetry has 16 supercharges. Thus, a generic multiplet is obtained by acting on a primary field with up to 16 supersymmetry transformations. The on-shell transformations are [17]

$$\begin{aligned}\delta A_{\alpha\dot{\alpha}} &= \bar{\eta}^{I\dot{\beta}}\epsilon_{\dot{\alpha}\dot{\beta}}\psi_{I\alpha} + \eta_I^\beta\epsilon_{\alpha\beta}\bar{\psi}_{\dot{\alpha}}^I, \\ \delta\phi_{[IJ]} &= \eta_{[I}^\alpha\psi_{J]\alpha} + \epsilon_{IJKL}\bar{\eta}^{K\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}^L, \\ \delta\psi_{I\alpha} &= \eta_I^\beta F_{(\alpha\beta)} + \eta_J^\beta\epsilon_{\alpha\beta}[\phi_{IK}, \phi^{JK}] + \bar{\eta}^{J\dot{\beta}}D_{\alpha\dot{\beta}}\phi_{IJ}, \\ \delta F_{(\alpha\beta)} &= \bar{\eta}^{I\dot{\gamma}}D_{\dot{\gamma}(\alpha}\psi_{\beta)I},\end{aligned}\tag{1.1}$$

where $\phi_{[IJ]}$, $\psi_{I\alpha}$, and $A_{\alpha\dot{\alpha}}$ are the scalar, fermion, and gauge components of the $\mathcal{N} = 4$ multiplet, respectively. $F_{(\alpha\beta)}$ denotes the gauge field strength tensor. All these fields transform in the adjoint representation of the gauge group, but we omitted the gauge index. Latin indices are the fundamental indices of $SU(4)_R$, while the Greek indices refer to the Lorentz group. Parameters η_I^α and $\bar{\eta}^{I\dot{\alpha}}$ all anticommute and denote the action of supercharges Q_α^I and $\bar{Q}_{I\dot{\alpha}}$.

A primary operator cannot, by definition, be expressed as a supersymmetry transformation of some other operator. Hence, primary operators are products of the scalar fields $\phi_{[IJ]}$, with color indices contracted without the use of the f symbol. Any antisymmetric contraction of gauge indices indicates that the operator contains some descendent because the supersymmetry transformations (1.1) contain a commutator of scalar fields.

Short or “chiral” multiplets are generated by at most eight supersymmetry transformations of a primary field, while the remaining transformations annihilate the field. In the notation of Eq. (1.1), a short multiplet is generated by at most four Q_α^I ’s (corresponding to η_I^α ’s) and no more than four $\bar{Q}_{I\dot{\alpha}}$ ($\bar{\eta}^{I\dot{\alpha}}$).

Primary gauge-invariant fields ϕ^k in the $(0, k, 0)$ representation of $SU(4)_R$ generate short multiplets. There are several arguments in the literature explaining that fact [18–20]. These arguments do not make any reference to the gauge structure of the primary fields. Short

multiplets were constructed using the oscillator method, in which the representations of non-compact groups are built from unitary representations of a compact subgroup [18, 19]. Moreover, representations that contain “null vectors” under the action of supersymmetry transformations are short [10, 20]. This argument also relies on the global symmetries of the primary field, and not its gauge structure.

Another way of showing that ϕ^k in $(0, k, 0)$ are in short multiplets is by considering the $\mathcal{N} = 1$ superconformal algebra. In the $\mathcal{N} = 1$ language, only $SU(3) \times U(1)$ subgroup of $SU(4)$ is manifest. Scalar fields are in the antisymmetric **(6)** representation, they decompose into chiral $\mathbf{3}_{2/3}$ and antichiral $\bar{\mathbf{3}}_{-2/3}$ of $SU(3) \times U(1)$. The decomposition of ϕ^k in $(0, k, 0)$ into $SU(3) \times U(1)$ involves a k -index symmetric tensor of $SU(3)$ carrying the R-charge of $\frac{2k}{3}$. Such a field must be $\mathcal{N} = 1$ chiral since it is a product of k scalar components of chiral superfields. This means that the original multiplet must be short and that its dimension is fixed by the $\mathcal{N} = 1$ superalgebra to be k .

2 Two- and Three-Point Correlators

Conformal symmetry uniquely specifies the form of two- and three-point correlator functions. Anomalous dimensions of operators show up in correlators as logarithmic corrections. Such logarithms arise as the result of expanding exponentials of the anomalous dimensions in the powers of coupling. In the case of protected operators, only the overall coefficient of a correlator function can depend on the coupling constant. We will show the absence of radiative corrections at $O(g^2)$ in any form—containing either logarithms or powers of the space-time points. This affirms that the operators we consider have protected dimensions.

We now show that two-point correlators of arbitrary linear combinations of operators in the $(0, k, 0)$ representation are not corrected at order g^2 . We denote the single-trace operators by

$$O_k = \text{Tr}(T^{a_1} \dots T^{a_k}) \phi_{I_1 J_1}^{a_1} \dots \phi_{I_k J_k}^{a_k},$$

while generic multi- or single-trace operators by

$$M_k^r = \text{Tr}(T^{a_1} \dots T^{a_{m_1}}) \dots \text{Tr}(T^{a_{m_r+1}} \dots T^{a_k}) \phi_{I_1 J_1}^{a_1} \dots \phi_{I_k J_k}^{a_k}.$$

The set of numbers m_1, \dots, m_r defines the number of generators in each of the r traces. The k -th trace contains $m_k - m_{k-1}$ generators, where $m_0 = 0$. With this definition $O_k = M_k^1$. In the above formulas, matrices T^a are the generators of any simple nonabelian gauge group. Indices a are the adjoint indices. The generators can be in any representation since we will only use the cyclic property of the trace and the Lie algebra relations about commutators.

Following Ref. [6] we work in the $\mathcal{N} = 1$ component language. We refer to the scalar components of chiral superfields as z^n and the antichiral ones as \bar{z}^n , where $n = 1, 2, 3$ is the $SU(3)$ index. The calculation we carry out here follows closely that of Ref. [6]. We will use

the same notation. We denote the propagator of scalar fields by

$$G(x, y) = \frac{1}{4\pi^2} \frac{1}{(x - y)^2}.$$

There are two types of interactions that enter the calculation of the two-point function. The first interaction is the self-energy correction to the scalar propagator

$$\text{---} \overset{a}{\bullet} \text{---} \overset{a'}{\bullet} = \delta^{aa'} C_2(G) A(x, y) G(x, y), \quad (2.1)$$

where a, a' are the gauge indices, and $C_2(G)$ is the Casimir of the adjoint representation of the gauge group G . We will not need the explicit form of function $A(x, y)$. Second, there are four-point interactions which come from summing the gluon exchange diagram and quartic interaction originating from the D-term. The sum of these two diagrams gives

$$\begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \text{---} \quad \text{---} \\ \diagup \quad \diagdown \\ a' \quad b' \end{array} + \begin{array}{c} \text{---} \quad \text{---} \\ \diagdown \quad \diagup \\ \bullet \\ \diagup \quad \diagdown \\ \text{---} \quad \text{---} \end{array} = (f^{pab} f^{pa'b'} + f^{pab'} f^{pa'b}) B(x, y) G^2(x, y), \quad (2.2)$$

where, again, the explicit form of $B(x, y)$ is irrelevant.

We always choose the highest-weight state in the k -index symmetric representation of $SU(3)$, in order to avoid complications with the combinatorics of flavor symmetry. For a two-point function this implies that one of the operators is proportional to k powers of z^1 , the other to \bar{z}^1 . The tree-level expression for the correlator of two, not necessary identical, operators M_k is given by

$$\begin{aligned} \langle M_k^{(1)}(x) M_k^{(2)}(y) \rangle &= G(x, y)^k P_{k,k,0}^{(1,2)}, \\ P_{k,k,0}^{(1,2)} &= \sum_{\text{perms } \sigma} [\text{Tr}(T^{a_1} \dots T^{a_{m_1}}) \dots \text{Tr}(T^{a_{m_r+1}} \dots T^{a_k})] \\ &\quad \cdot \left[\text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(m'_1)}}) \dots \text{Tr}(T^{a_{\sigma(m'_r+1)}} \dots T^{a_{\sigma(k)}}) \right]. \end{aligned} \quad (2.3)$$

The self-energy graphs are trivial to incorporate: we need to multiply one of the propagators by $C_2(G)A(x, y)$ and then sum over all propagators. This sum is equal to

$$k C_2(G) A(x, y) G(x, y)^k P_{k,k,0}^{(1,2)}. \quad (2.4)$$

We now insert the interaction of Eq. (2.2) into the tree-level expression (2.3) and sum over all pairs of lines. The f symbols are converted into commutators on one of the multi-trace expressions. We do not repeat the whole expression (2.3) with additional commutators inserted, but instead write only the relevant part

$$\sum_{i \neq j=1}^k \{ \text{Tr}(T^{a_1} \dots [T^{a_i}, T^p] \dots T^{a_{m_1}}) \dots \text{Tr}(\dots [T^{a_j}, T^p] \dots) \dots \text{Tr}(T^{a_{m_r+1}} \dots T^{a_k}) \}. \quad (2.5)$$

The sum over all pairs of indices can be expressed as

$$\sum_{i \neq j=1}^k = \sum_{i,j \in \text{different traces}} + \sum_{\text{each trace}} \sum_{i \neq j \in \text{same trace}} \quad (2.6)$$

We now use the identity

$$\sum_{i=1}^k (M_1 \dots [M_i, N] \dots M_k) = 0 \quad (2.7)$$

to observe that

$$\sum_{i,j \in \text{different traces}} \{\text{Tr}(T^{a_1} \dots [T^{a_i}, T^p] \dots T^{a_{m_1}}) \dots \text{Tr}(\dots [T^{a_j}, T^p] \dots) \dots\} = 0. \quad (2.8)$$

This is the observation that allows us to extend the results of Ref. [6] from single traces to multiple traces. We will use similar identities repeatedly. The remaining sum in Eq. (2.6) can be performed for each trace individually. We can again use Eq. (2.7) to rewrite the sums within a given trace

$$\begin{aligned} \sum_{i \neq j=1}^{k_i} \text{Tr}(\dots [T^i, T^p] \dots [T^j, T^p] \dots) = \\ - \sum_{i=1}^{k_i} \text{Tr}(\dots [[T^i, T^p], T^p] \dots) = -k_i C_2(G) \text{Tr}(\dots), \end{aligned} \quad (2.9)$$

where k_i is the number of generators in the trace. It is now easy to perform the sum over each trace, which gives

$$\frac{1}{2} k B(x, y) G(x, y)^k P_{k,k,0}^{(1,2)}, \quad (2.10)$$

where the extra factor of $1/2$ is a combinatoric factor that prevents overcounting of diagrams. The additional minus sign comes from converting two f symbols into commutators. It is now apparent that the cancellation of radiative corrections happens for the same reason it does in the two-point function of single traces. It was shown in Ref. [6] that $2A + B = 0$ due to a non-renormalization theorem for $\langle O_2 O_2 \rangle$, which is a consequence of O_2 being in the multiplet containing the energy-momentum tensor.

Similarly, we can repeat the analysis for the $\langle M_k(x) M'_k(y) O_2(w) \rangle$ correlator. Calculation for the general three-point functions will be presented in the Appendix. In addition to $O(g^2)$ interactions of Eqs. (2.1) and (2.2) there is a four-scalar interaction involving three

space-time points. This can be illustrated as

$$\begin{array}{c} \text{a} \\ \text{b}^y \\ \text{a}' \\ \text{b}'_w \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} = (f^{pab} f^{pa'b'} + f^{pab'} f^{pa'b}) C(x; y, w) G(x, y) G(x, w), \quad (2.11)$$

where the form of function C will not be important for the calculation. In some cases only the D -term quartic interaction contributes in the above diagram defining C . Whenever both F - and D -terms contribute, we denote the corresponding function by C' .

The tree-level expression for this correlator is

$$\begin{aligned} \langle M_k^{(1)}(x) M_k^{(2)}(y) O_2(w) \rangle &= G(x, y)^{k-1} G(x, w) G(y, w) P_{k,k,2}^{(1,2)} t_{11}, \\ P_{k,k,2}^{(1,2)} &= k P_{k,k,2}^{(1,2)}, \end{aligned} \quad (2.12)$$

where t_{ij} is an $SU(3)$ flavor matrix. This matrix enters our definition of $O_2 = \bar{z}^i t_{ij} z^j$. In the above equation $M_k^{(1)} = (z^1)^k$ and $M_k^{(2)} = (\bar{z}^1)^k$. The additional factor of k comes from distinguishing one line in each multi-trace expression. It is straightforward to sum all possible self-energy diagrams since they multiply each propagator by the same expression. The sum is proportional to the tree-level term, which we omit, times the following factor

$$C_2(G) [(k-1)A(x, y) + A(x, w) + A(y, w)]. \quad (2.13)$$

Next, we include quartic interactions (2.2) by summing over all pairs of $k-1$ propagators that connect the operators $M_k^{(1)}(x)$ and $M_k^{(2)}(y)$. Repeating the same steps we have already explained for the two-point function, we convert the f symbols into commutators and then perform the sum over pairs of indices. Since there is only one distinguished generator in the traces at points x and y , Eq. (2.8) still holds. The distinguished generator is the one connected by a propagator to the operator O_2 . One can always use the identity (2.7) for a trace that does not contain the distinguished line. The remaining sum in Eq. (2.6) is easy to carry out. For the traces without the distinguished generator we pick a factor of $-k_i$, which is the negative of the number of elements in a given trace, just like we did in Eq. (2.9). For the trace with the distinguished generator, the algebra is slightly more involved. We can always use the cyclic property of the trace to place the special generator as the last one in the trace. Then, we use the following identity

$$\begin{aligned} & \sum_{i \neq j=1}^{k_i-1} \text{Tr}(\dots [T^{a_i}, T^p] \dots [T^{a_j}, T^p] \dots) \\ &= - \sum_{i=1}^{k_i-1} \{ \text{Tr}(\dots [[T^{a_i}, T^p], T^p] \dots) + \text{Tr}(\dots [T^{a_i}, T^p] \dots [T^{a_{k_i}}, T^p]) \} \\ &= -(k_i - 1 - 1) C_2(G) \text{Tr}(\dots) \end{aligned} \quad (2.14)$$

to complete the evaluation of the sum over pairs of propagators between the two operators M_k 's. There are $k - 1$ such propagators. Due to the additional -1 in Eq. (2.14) the final expression is proportional to $k - 2$.

There are three possible ways of inserting interaction (2.11) into the tree-level expression, depending on which vertex the two lines grouped together enter. This distinguished vertex in Eq. (2.11) corresponds to the space-time point x . When the distinguished vertex is one of the vertices corresponding to operators $M_k^{(1)}(x)$ or $M_k^{(2)}(y)$, we need to sum over all propagators connecting these vertices. This single sum is easy to perform using the identity (2.7) since the sum over all, but one, traces vanishes. The selected trace is the one which involves a generator linked to the operator $O_2(w)$ by the scalar propagator. The algebra is identical to the one we used in the previous equation, which is

$$\sum_{i=1}^{k_i-1} \text{Tr}(\dots [T^{a_i}, T^p] \dots [T^{a_{k_i}}, T^p]) = -C_2(G) \text{Tr}(T^{a_1} \dots T^{a_{k_i}}). \quad (2.15)$$

When Eq. (2.11) is inserted with the distinguished vertex corresponding to the operator $O_2(w)$ there is no need to perform any sums since there are only two propagators entering this vertex. However, the choice of flavors at this vertex is such that both the D -term and F -term vertices contribute in this case. Thus, the expression is proportional to $C'(w; x, y)$ instead of C . The sum of all $O(g^2)$ contributions is equal to the Born term times

$$C_2(G) \left[(k-1)A(x, y) + A(x, w) + A(y, w) + \frac{k-2}{2}B(x, y) + 2C(x; y, w) + 2C(y; x, w) + 2C'(w; x, y) \right]. \quad (2.16)$$

The minus sign in front of functions C compared to Eq. (2.15) is the result of converting f symbols into commutators. The factor of two comes from two pairs of f symbols in the definition (2.11) of function C . Eq. (2.16) is equal to zero as a result of non-renormalization of the $\langle O_2(x)O_2(y)O_2(w) \rangle$ correlator, which implies that [6]

$$A(x, y) + A(x, w) + A(y, w) + 2[C(x; y, w) + C(y; x, w) + C'(w; x, y)] = 0. \quad (2.17)$$

This equation together with the fact that $2A + B = 0$ imply the lack of $O(g^2)$ corrections to the $\langle M_k(x)M'_k(y)O_2(w) \rangle$ correlator. In the Appendix, we show that $O(g^2)$ corrections for general three-point functions $\langle M_{k_1}M_{k_2}M_{k_3} \rangle$ vanish.

3 Conclusions

We investigated correlators of primary fields of short multi-trace operators in $\mathcal{N} = 4$ supersymmetry. Such primary fields contain k scalar fields in the $(0, k, 0)$ representation of $SU(4)_R$

and arbitrary symmetric contraction of color indices. We have shown that two- and three-point functions of chiral primary fields do not receive radiative corrections at order g^2 in perturbation theory. This result is valid for any gauge group due to the non-renormalization theorem for two- and three-point correlators of the so-called “massless” multiplet, which includes the energy-momentum tensor. The result is also independent of the size of the gauge group, so it is not a leading N result in the large- N expansion.

Our results suggest that two- and three-point correlators of arbitrary short multiplets are not renormalized. If these results hold beyond low orders in perturbation theory, it indicates that there must be a symmetry reason for such non-renormalization. The so-called bonus $U(1)_Y$ symmetry [12] explains why correlators of operators which correspond to single-particle states of supergravity on AdS spaces are not renormalized. Perhaps it is possible to extend the arguments about the $U(1)_Y$ symmetry to correlators of short operators corresponding to multi-particle states. Since all the quantum numbers of the single- and multi-trace chiral primary operators are identical, non-renormalization theorems are likely to be a consequence of the full $\mathcal{N} = 4$ superconformal symmetry. Superconformal Ward identities would not distinguish between single-trace and multi-trace operators.

It would be interesting to explore the implications of these results for the duality between supergravity and conformal field theory. One of the issues that require better understanding is the correspondence between operators in the conformal field theory and supergravity states. It is believed that single-particle states of supergravity correspond to short single-trace operators. However, the single- and multi-trace operators mix. It is easy to check that at the tree-level, the two-point functions $\langle O_k M_k \rangle$ do not vanish. In the $SU(N)$ theory, two-point functions of single-trace operators O_k are proportional to N^k in the leading order in N . Likewise, in the leading order $\langle O_k M_k^r \rangle \propto N^{k-r+1}$, where r is the number of color traces in the operator M_k^r . Thus, the mixing of operators O_k with M_k ’s is suppressed in the large- N limit.

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Appendix $\langle M_{k_1} M_{k_2} M_{k_3} \rangle$

We now extend the calculations of Section 2. We show that arbitrary three-point correlators $\langle M_{k_1}(x) M_{k_2}(y) M_{k_3}(w) \rangle$ are not corrected at order g^2 . We define the sum over all

permutations of indices in a given multi-trace expression as

$$M_{k_i} \propto \Sigma \text{Tr}(T^{a_1} \dots T^{a_{k_i}}) \equiv \frac{1}{k_i!} \sum_{\text{perms } \sigma} \text{Tr}(T^{a_{\sigma(1)}} \dots T^{a_{\sigma(m_1)}}) \dots \text{Tr}(T^{a_{\sigma(m_r+1)}} \dots T^{a_{\sigma(k_i)}}), \quad (\text{A.1})$$

where the number of traces and the number of generators in each trace depends on the gauge structure of a particular operator. The number of propagators in the tree-level expression that connect operators with k_1 and k_2 fields are $\alpha_3 = (k_1 + k_2 - k_3)/2$ and so on. As in the previous sections, $M_{k_1} \propto (z^1)^{k_1}$ is $\mathcal{N} = 1$ chiral and it is the highest-weight field of the completely symmetric $SU(3)$ representation. Similarly, $M_{k_2} \propto (\bar{z}^1)^{k_2}$ is antichiral, while $M_{k_3} \propto (z^1)^{\alpha_1} (\bar{z}^1)^{\alpha_2}$.

With the above definitions, the tree-level expression for the correlator is

$$\begin{aligned} \langle M_{k_1}(x) M_{k_2}(y) M_{k_3}(w) \rangle &\propto G(x, y)^{\alpha_3} G(x, w)^{\alpha_2} G(y, w)^{\alpha_1} P_{k_1, k_2, k_3}, \\ P_{k_1, k_2, k_3} &= \Sigma \text{Tr}(T^{a_1} \dots T^{a_{\alpha_3}} T^{b_1} \dots T^{b_{\alpha_2}}) \Sigma \text{Tr}(T^{b_1} \dots T^{b_{\alpha_2}} T^{c_1} \dots T^{c_{\alpha_1}}) \\ &\quad \cdot \Sigma \text{Tr}(T^{c_1} \dots T^{c_{\alpha_1}} T^{a_1} \dots T^{a_{\alpha_3}}). \end{aligned} \quad (\text{A.2})$$

In the above formula we omitted the $SU(3)$ flavor generator, which contracts the flavor indices of chiral and antichiral fields in $M_{k_3}(w)$. The computation of radiative corrections is very similar to the one we presented for the $\langle M_k M'_k O_2 \rangle$ correlator, only the algebra of traces is more complicated. There are three types of diagrams that contribute to the $O(g^2)$ corrections. These correspond to insertions of $O(g^2)$ interactions of Eqs. (2.1), (2.2), and (2.11) into the tree-level expression for the correlator.

The self-energy graphs are straightforward to incorporate. They give a factor of $C_2(G)A$ for every propagator, so we get

$$C_2(G) [\alpha_3 A(x, y) + \alpha_2 A(x, w) + \alpha_1 A(y, w)] \quad (\text{A.3})$$

times the Born expression. Next, we include the interactions of Eq. (2.11). For each of the three vertices we need to evaluate

$$W_{k_1, k_2, k_3} \equiv \sum_{i=1}^{\alpha_3} \sum_{j=1}^{\alpha_2} \Sigma \text{Tr}(T^{a_1} \dots [T^{a_i}, T^p] \dots T^{a_{\alpha_3}} T^{b_1} \dots [T^{b_j}, T^p] \dots T^{b_{\alpha_2}}) \Sigma \text{Tr}(\dots) \Sigma \text{Tr}(\dots). \quad (\text{A.4})$$

However, it turns out that all three such expressions are identical. We can do the algebra for a fixed, yet sufficiently general, permutation of the generators. Performing the sums over all permutations will not affect the result. We divide the traces in the multi-trace expression $\Sigma \text{Tr}(T^{a_1} \dots T^{b_{\alpha_3}})$ into three categories: traces containing only generators with index a , only index b , and containing both indices. For simplicity, we choose the permutations of indices such that for traces with two different indices, the generators with index a come before the

ones with index b . For example,

$$\begin{aligned} & \text{Tr}(T^a \dots T^a) \dots \text{Tr}(T^a \dots T^a) \text{Tr}(T^a \dots T^a T^b \dots T^b) \dots \text{Tr}(T^a \dots T^a T^b \dots T^b) \\ & \cdot \text{Tr}(T^b \dots T^b) \dots \text{Tr}(T^b \dots T^b). \end{aligned} \quad (\text{A.5})$$

We choose analogous permutations for the other two traces. When performing the sums $\sum_{i=1}^{\alpha_3}$ and $\sum_{j=1}^{\alpha_2}$ in Eq. (A.5), we can neglect all traces with one kind of indices only. Sums over these traces vanish due to Eq. (2.7). Our goal is to show that we can move the sums from one multi-trace expression to another. Converting the commutators into the f symbols and then back into commutators on another trace, we get

$$\begin{aligned} W_{k_1, k_2, k_3} &= \Sigma \text{Tr}(T^{a_1} \dots T^{b_{\alpha_3}}) \sum_{j=1}^{\alpha_2} \Sigma \text{Tr}(T^{b_1} \dots [T^{b_j}, T^p] \dots T^{b_{\alpha_2}} T^{c_1} \dots T^{c_{\alpha_1}}) \\ & \cdot \sum_{i=1}^{\alpha_3} \Sigma \text{Tr}(T^{c_1} \dots T^{c_{\alpha_1}} T^{a_1} \dots [T^{a_i}, T^p] \dots T^{a_{\alpha_3}}). \end{aligned} \quad (\text{A.6})$$

Within a given multi-trace expression we can convert a sum over one kind of index into minus the sum over another index

$$\begin{aligned} & \sum_{i=1}^{\alpha_3} \text{Tr}(T^{a_1} \dots [T^{a_i}, T^p] \dots T^a T^b \dots T^b) \dots \text{Tr}(T^a \dots T^{a_{\alpha_3}} T^b \dots T^b) \\ &= - \sum_{j=1}^{\alpha_2} \text{Tr}(T^a \dots T^a T^{b_1} \dots [T^{b_j}, T^p] \dots T^b) \dots \text{Tr}(T^a \dots T^a T^b \dots T^{b_{\alpha_2}}) \end{aligned} \quad (\text{A.7})$$

because we can use Eq. (2.7) within each trace independently. In Eq. (A.6) we can exchange the sum over b 's into the sum over c 's using the above trick. We can then move the sum over c 's into the last trace in Eq. (A.6) by converting f symbols into traces. This shows that the expression defined in Eq. (A.4) is universal for all insertions of interactions (2.11), hence we obtain

$$-2W_{k_1, k_2, k_3} [C(x; y, w) + C(y; x, w) + C'(w; x, y)]. \quad (\text{A.8})$$

The overall factor of -2 is the result of converting two pairs of f symbols into propagators. This is the same factor as the one appearing in Eq. (2.16).

The final step is evaluating the contributions of interactions of Eq. (2.2). For each set of indices a , b , and c , we need to compute the double sum

$$\begin{aligned} & \sum_{i \neq j=1}^{\alpha_3} \Sigma \text{Tr}(T^{a_1} \dots [T^{a_i}, T^p] \dots [T^{a_j}, T^p] \dots T^{a_{\alpha_3}} T^{b_1} \dots T^{b_{\alpha_2}}) \Sigma \text{Tr}(\dots) \Sigma \text{Tr}(\dots) \\ &= - \left[\sum_{i=1}^{\alpha_3} \sum_{j=1}^{\alpha_2} \Sigma \text{Tr}(T^{a_1} \dots [T^{a_i}, T^p] \dots T^{a_{\alpha_3}} T^{b_1} \dots [T^{b_j}, T^p] \dots T^{b_{\alpha_2}}) \right] \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{\alpha_3} (T^{a_1} \dots [[T^{a_i}, T^p], T^p] \dots T^{b_1} \dots T^{b_{\alpha_2}}) \Big] \Sigma \text{Tr}(\dots) \Sigma \text{Tr}(\dots) \\
& = -C_2(G) \alpha_3 P_{k_1, k_2, k_3} - W_{k_1, k_2, k_3},
\end{aligned} \tag{A.9}$$

where the first equality follows from applying Eq. (2.7) to a fixed permutation of indices, just like we did in Eq. (A.5). Therefore, the contribution of interactions from Eq. (2.2) is proportional to

$$\begin{aligned}
& \frac{1}{2} C_2(G) P_{k_1, k_2, k_3} [\alpha_3 B(x, y) + \alpha_2 B(x, w) + \alpha_1 B(y, w)] \\
& + \frac{1}{2} W_{k_1, k_2, k_3} [B(x, y) + B(x, w) + B(y, w)],
\end{aligned} \tag{A.10}$$

where the factor of $\frac{-1}{2}$ has the same origin as in Eq. (2.10). The sum of all $O(g^2)$ interactions presented in Eqs. (A.3), (A.8), and (A.10) is

$$\begin{aligned}
& W_{k_1, k_2, k_3} \left\{ -2 [C(x; y, w) + C(y; x, w) + C'(w; x, y)] + \frac{1}{2} [B(x, y) + B(x, w) + B(y, w)] \right\} \\
& + C_2(G) P_{k_1, k_2, k_3} \left\{ \alpha_3 (A + \frac{1}{2} B)(x, y) + \alpha_2 (A + \frac{1}{2} B)(x, w) + \alpha_1 (A + \frac{1}{2} B)(y, w) \right\},
\end{aligned} \tag{A.11}$$

where the first line vanishes after using both $2A + B = 0$ and Eq. (2.17).

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